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MAXIMAL AVERAGE ALONG VARIABLE LINES*

BY

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ABSTRACT

We prove the L^p boundedness of the maximal operator associated with a family of lines $l_x = \{(x_1, x_2) - t(1, a(x_1)) : t \in [0, \infty)\}$ when a' is a positive increasing function.

1. Introduction

Let $v : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field. We define a maximal function along a family of lines $\{x - tv(x) : t \in \mathbb{R}\}$ by

$$Mf(x) = \sup_{r>0} \frac{1}{r} \int_0^r |f(x - tv(x))| dt, \tag{1.1}$$

where $x = (x_1, x_2)$. We assume that our vector field v depends on only one variable x_1 given by

$$v(x_1, x_2) = (1, a(x_1)),$$

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where a is a real valued function defined on $[x_0,\infty)$. Consider the maximal operator

$$\mathcal{M}_a f(x) = \chi_{[x_0,\infty)}(x_1) \sup_{r>0} \frac{1}{r} \int_0^r |f(x_1 - t, x_2 - a(x_1)t)| dt. \tag{1.2}$$

In [2], Carbery, Seeger, Wainger and Wright study the maximal operator $\mathcal{M}_a$ and the corresponding Hilbert transform. They obtain the L^p boundedness of those operators under the convexity assumption of a and the additional size conditions of a and a' such that

$$0 < c \leq \left| \frac{a'(t)|I_l|}{a(t)} \right| \leq C \quad \text{for } t \in I_l = \{u : 2^l \leq a'(u) \leq 2^{l+1}\}, \tag{1.3}$$

where C and c are constants independent of l. In this paper we prove the L^p-boundedness of the maximal operator $\mathcal{M}_a$ without the assumption of (1.3).

Theorem 1: *Suppose that $a' : [t_0,\infty) \to [0,\infty)$ is a monotonic increasing function such that $a'(t_0) = 0$. Then $\mathcal{M}_a$ is bounded on $L^p(\mathbb{R}^2)$ for $1 < p \leq \infty$.*

Remark:

(1) Let v in (1.1) be a Lipschitz vector field from $\mathbb{R}^2$ into the unit circle S^1. By A. Zygmund, it is conjectured that the average of the L^2-function along the line segment $\{x - tv(x) : t \in (-r,r)\}$ converges to itself for almost every x as r approaches to zero. It is still an open problem and seems to be unknown even if the Lipschitz condition is replaced by the smoothness condition on v as proposed in [6].

(2) If v is real analytic or satisfies a certain finite type condition, then the local version of M is bounded in L^p with $p > 1$, see [1] and [3].

(3) In [4], N. Katz has made an interesting weak type (2,2) bound of M when v is appropriately restricted to a finite number of points in S^1.

(4) As a singular integral version of the Zygmund conjecture, E. M. Stein conjectured that the Hilbert transform H_v along the line

$$\{x - tv(x) : -\varepsilon \leq t \leq \varepsilon\}$$

is of weak type (2.2) under the Lipschitz condition of v. M. Lacey and X. Li [5] introduced a new object called a Lipschitz Kakeya maximal function. They proved that Steins conjecture is resolved if the Lipschitz Kakeya maximal function is shown to be bounded in L^p for some

$1 < p < 2$. They obtained the weak type (2.2) estimate for the Lipschitz Kakeya maximal function.

2. Sketch of Proof

Let $t_1 = \sup\{t : a'(t) = 0\}$. Then, for $x_1 \in [t_0, t_1]$, $\mathcal{M}_a f(x)$ is controlled by the one dimensional Hardy-Littlewood maximal function. Thus we may assume that $a'(x_1) > 0$. Let $\varphi_j(y) = \varphi(y/2^j)/2^j$ with $\varphi \in C_0^\infty(1/2, 1)$. By the diadic decomposition of the interval $[0, r]$, we define a diadic version of $\mathcal{M}_a$ by

$$\mathfrak{M}_a f(x) = \sup_{j \in \mathbb{Z}} |M_j f(x)|,$$

where

$$M_j f(x) = \int f(x_1 - t, x_2 - a(x_1)t)\varphi_j(t)dt.$$

Then it suffices to prove the $L^p(\mathbb{R}^2)$ boundedness of $\mathfrak{M}_a$ in proving Theorem 1. By using

$$\delta(x_2 - y_2 - a(x_1)(x_1 - y_1)) = \int e^{i\lambda[x_2 - y_2 - a(x_1)(x_1 - y_1)]} d\lambda,$$

we rewrite

$$M_j f(x) = \iiint e^{i\lambda[x_2 - y_2 - a(x_1)(x_1 - y_1)]} \varphi_j(x_1 - y_1) f(y_1, y_2) dy_1 dy_2 d\lambda.$$

Let us choose an even function $\psi \in C_0^\infty(-1, 1)$ such that $\psi \equiv 1$ on $[-1/2, 1/2]$. Let $\chi(\xi) = \psi(\xi/2) - \psi(\xi)$. Then for each fixed j and $\lambda \neq 0$,

$$1 = \sum_{n=-\infty}^{\infty} \chi(\lambda a'(x_1) 2^{2j+n}) = \psi(\lambda a'(x_1) 2^{2j}) + \sum_{n=-\infty}^{n=0} \chi(\lambda a'(x_1) 2^{2j+n}).$$

According to the size $|\lambda a'(x_1) 2^{2j}|$, we decompose

$$M_j f(x) = M_j^{loc} f(x) + \sum_{n=-\infty}^{n=0} M_j^n f(x)$$

where

$$M_j^{loc} f(x) = \iiint e^{i\lambda[x_2 - y_2 - a(x_1)(x_1 - y_1)]} \varphi_j(x_1 - y_1) \psi(\lambda a'(x_1) 2^{2j}) f(y) dy d\lambda,$$

$$M_j^n f(x) = \iiint e^{i\lambda[x_2 - y_2 - a(x_1)(x_1 - y_1)]} \varphi_j(x_1 - y_1) \chi(\lambda a'(x_1) 2^{2j+n}) f(y) dy d\lambda.$$

Thus our maximal function $\mathfrak{M}_a f$ is majorized by $\mathfrak{M}^{loc}f + \mathfrak{M}^{glo}f$ where

$$\mathfrak{M}^{loc}f(x) = \sup_{j\in\mathbb{Z}} |M_j^{loc}f(x)|, \tag{2.1}$$

$$\mathfrak{M}^{glo}f(x) = \sum_{n=-\infty}^{n=0} \sup_{j\in\mathbb{Z}} |M_j^n f(x)|. \tag{2.2}$$

For the local part estimate $\mathfrak{M}^{loc}$, we give a weak type (1,1) estimate by making an appropriate Vitali-type covering lemma for a family of variable parallelograms. For this, we need only the condition that a' is increasing function. This will be established in Section 3.

For the global part $\mathfrak{M}^{glo}$, we obtain the L^2 bound by estimating

$$\|M_j^n[M_j^n]^*\|_{L^2\to L^2} = O(2^{-c|n|}), \quad \|M_{j_1}^n[M_{j_2}^n]^*\|_{L^2\to L^2} = O(2^{-c|j_1-j_2|}),$$

for some $c > 0$. To derive the L^p theory, we use the bootstrap argument combined with the Littlewood-Paley decomposition on the second coordinate of the frequency variable. This will be done in Section 4.

In [2], the authors handled each piece M_j by comparing the sizes of 2^j and $|I_l|$, where $I_l = \{u : 2^l \le a'(u) \le 2^{l+1}\}$. The case $2^j < |I_l|$ was reduced to the case having nonvanishing rotational curvature by localization argument. The comparability condition (1.3) was used to make the uniform lower bound of the rotational curvature. For the case $2^j > |I_l|$, the condition (1.3) is used to control the size of the curvature $|\lambda a'(x_1)2^{2j}|$ in the almost orthogonality estimates. However we note that our main estimate is independent of the size $|I_l|$.

3. Weak type (1,1) estimates for $\mathfrak{M}^{loc}$

For each $x \in \mathbb{R}^2$ and $j \in \mathbb{Z}$, we set

$$H_x(j) = \{y : 2^{j-1} < x_1 - y_1 < 2^j, |x_2 - y_2 - a(x_1)(x_1 - y_1)| < 2^{2j}a'(x_1)\}.$$

Define the maximal operator associated with the class of the above parallelograms by

$$\mathcal{N}f(x) = \sup_{j\in\mathbb{Z}} \frac{1}{|H_x(j)|}\int_{H_x(j)} |f(y)|dy. \tag{3.1}$$

Let us rewrite $M_j^{loc}f(x)$ as

$$\iint \varphi_j(x_1 - y_1)\frac{1}{2^{2j}a'(x_1)}\widehat{\psi}\left(\frac{x_2 - y_2 - a(x_1)(x_1 - y_1)}{2^{2j}a'(x_1)}\right)f(y_1, y_2)dy. \tag{3.2}$$

Since $\widehat{\psi}$ in (3.2) is a rapidly decreasing function, we observe that

$$\mathfrak{M}^{loc} f(x) \ \leq C\mathcal{N} f(x).$$

Thus the weak type (1,1)-boundedness of $\mathfrak{M}^{loc}$ will be proved by the following estimates:

PROPOSITION 1: *Suppose that a' is a positive increasing function, then $\mathcal{N}$ is weak type* $(1,1)$.

For the proof, we make a variant of Vitali-covering lemma corresponding to the parallelogram $H_x(j)$'s. Let us define two sets $B_x(j)$ and $B_x^*(j)$ associated with $H_x(j)$ such that

$$B_x(j) = \{y : |x_1 - y_1| < 2^j, |x_2 - y_2 - a(x_1)(x_1 - y_1)| < 2^{2j} a'(x_1)\},$$
$$B_x^*(j) = \{y : |x_1 - y_1| < 2^{j+3}, |x_2 - y_2 - a(x_1)(x_1 - y_1)| < 2^{2(j+5))} a'(x_1)\}.$$

Then note that $H_x(j) \subset B_x(j) \subset B_x^*(j)$ and the size of each set is comparable with the others

$$\begin{aligned} |H_x(j)| &= a'(x_1) 2^{3j}, \\ |B_x(j)| &= a'(x_1) 2^{3j+2}, \\ |B_x^*(j)| &= a'(x_1) 2^{3(j+5)}. \end{aligned} \tag{3.3}$$

With the above size information we need an engulfing property which is crucial in the Vitali-type covering lemma.

LEMMA 1: *Let j be an integer valued function on $\mathbb{R}^2$ and a' be a positive increasing function. If $|H_z(j(z))| \leq |H_x(j(x))|$ and $H_z(j(z)) \cap H_x(j(x)) \neq \emptyset$, then $B_z(j(z)) \subset B_x^*(j(x))$.*

Proof. Suppose that $x_1 \leq z_1$. Then $a'(x_1) \leq a'(z_1)$, since a' is an increasing function. Thus we obtain that $j(z) \leq j(x)$, in view of the inequality

$$2^{3j(z)} a'(z_1) = |H_z(j(z))| \leq |H_x(j(x))| = 2^{3j(x)} a'(x_1).$$

Since there exists $y \in H_z(j(z)) \cap H_x(j(x))$,

$$2^{j(z)} > z_1 - y_1 = (z_1 - x_1) + (x_1 - y_1) > 0 + 2^{j(x)-1}.$$

Thus it follows that $j(z) = j(x)$. From this and $|H_z(j(z))| \leq |H_x(j(x))|$, it also follows that $a'(z_1) \leq a'(x_1)$. So we have

$$a'(z_1) = a'(x_1) \quad \text{and} \quad j(z) = j(x) \quad \text{if} \quad x_1 \leq z_1. \tag{3.4}$$

Suppose that $z_1 \le x_1$. For $y \in H_z(j(z)) \cap H_x(j(x))$, we observe that

$$2^{j(x)} > x_1 - y_1 = (x_1 - z_1) + (z_1 - y_1) \ge z_1 - y_1 > 2^{j(z)-1}.$$

Therefore,

$$a'(z_1) \le a'(x_1) \quad \text{and} \quad j(z) \le j(x) \quad \text{if} \quad z_1 \le x_1. \tag{3.5}$$

Now we show that $B_z(j(z)) \subset B_x^*(j(x))$. Let us take $w \in B_z(j(z))$. By the hypothesis, there exists $y \in H_z(j(z)) \cap H_x(j(x))$. Thus we use the definition of $B_z(j)$ and $H_z(j)$ to obtain that

$$\begin{aligned} -2^{j(z)} < z_1 - w_1 < 2^{j(z)} \quad &\text{and} \quad |S(z,w)| < 2^{2j(z)} a'(z_1), \\ 2^{j(z)-1} < z_1 - y_1 < 2^{j(z)} \quad &\text{and} \quad |S(z,y)| < 2^{2j(z)} a'(z_1), \\ 2^{j(x)-1} < x_1 - y_1 < 2^{j(x)} \quad &\text{and} \quad |S(x,y)| < 2^{2j(x)} a'(x_1), \end{aligned} \tag{3.6}$$

where $S(u,v) = u_2 - v_2 - a(u_1)(u_1 - v_1)$ for $u, v \in \mathbb{R}^2$. From (3.4)–(3.6),

$$|x_1 - w_1| \le |x_1 - y_1| + |y_1 - z_1| + |z_1 - w_1| < 2^{j(x)+3}. \tag{3.7}$$

From (3.4)–(3.6),

$$\begin{aligned} |S(x,w)| &= |S(z,w) - S(z,y) + S(x,y) + (a(x_1) - a(z_1))(y_1 - w_1)| \\ &< a'(x_1) 2^{2(j(x)+3)} + \left| \int_{z_1}^{x_1} a'(t)dt(y_1 - w_1) \right| \\ &< a'(x_1) 2^{2(j(x)+4)}. \end{aligned} \tag{3.8}$$

Therefore from (3.7) and (3.8), $w \in B_x^*(j(x))$. ∎

By using Lemma 1 we can prove a variant of the Vitali-covering lemma:

Lemma 2: *Suppose that the hypothesis of the previous lemma is true. Let $\{B_{x_\alpha}(j(x_\alpha))\}_{\alpha=1}^N$ be a class of N parallelograms. Then there exists a subsequence $\{x_{\alpha_k}\}_{k=1}^M$ of $\{x_\alpha\}_{\alpha=1}^N$ satisfying*

$$\bigcup_{\alpha=1}^{\alpha=N} B_{x_\alpha}(j(x_\alpha)) \subset \bigcup_{k=1}^{\nu=M} B^*_{x_{\alpha_k}}(j(x_{\alpha_k})), \tag{3.9}$$

$$H_{x_{\alpha_k}}(j(x_{\alpha_k})) \cap H_{x_{\alpha_l}}(j(x_{\alpha_l})) = \emptyset \quad \text{if } k \ne l. \tag{3.10}$$

Proof. Choose $B_{x_{\alpha_1}}(j(x_{\alpha_1}))$ from $\mathcal{B}_1 = \{B_{x_\alpha}(j(x_\alpha))\}_{\alpha=1}^N$ so that $|H_{x_{\alpha_1}}(j(x_{\alpha_1}))|$ is one of the largest among $\mathcal{H}_1 = \{H_{x_\alpha}(j(x_\alpha))\}_{\alpha=1}^N$. For $m \ge 2$, we select $B_{x_{\alpha_m}}(j(x_{\alpha_m}))$ from the class

$$\mathcal{B}_m = \mathcal{B}_{m-1} - \{B_{x_\alpha}(j(x_\alpha)) : H_{x_\alpha}(j(x_\alpha)) \cap H_{x_{\alpha_{m-1}}}(j(x_{\alpha_{m-1}})) \ne \emptyset\}$$

so that $|H_{x_{\alpha_m}}(j(x_{\alpha_m}))|$ is one of the largest among the class

$$\mathcal{H}_m = \mathcal{H}_{m-1} - \{H_{x_\alpha}(j(x_\alpha)) : H_{x_\alpha}(j(x_\alpha)) \cap H_{x_{\alpha_{m-1}}}(j(x_{\alpha_{m-1}})) \neq \emptyset\}.$$

Our selection is done in M (which is less then equal to N) steps. Take any $B_{x_\alpha}(j(x_\alpha))$ with $1 \leq \alpha \leq N$. Then $H_{x_\alpha}(j(x_\alpha))$ meets some $H_{x_{\alpha_m}}(j(x_{\alpha_m}))$ and $|H_{x_\alpha}(j(x_\alpha))| \leq |H_{x_{\alpha_m}}(j(x_{\alpha_m}))|$, otherwise it would be a candidate of $M+1$-th step. By Lemma 1 or Lemma 2, $B_{x_\alpha}(j(x_\alpha)) \subset B^*_{x_{\alpha_m}}(j(x_{\alpha_m}))$. This proves (3.9). In each step we have chosen $H_{x_{\alpha_m}}(j(x_{\alpha_m}))$ satisfying (3.10). ∎

Proof of Proposition 1. Let D be a compact set contained in $\{x : \mathcal{N}f(x) > \lambda\}$. We show that

$$|D| \leq \frac{C}{\lambda}\|f\|_{L^1(\mathbb{R}^2)}.$$

By (3.1), for each $x \in D$ there exist $j(x) \in \mathbb{Z}$ such that

$$\lambda < \frac{1}{|H_x(j(x))|}\int_{H_x(j(x))} |f(y)|dy. \tag{3.11}$$

Note that $D \subset \bigcup_{x\in D} B_x(j(x))$ where $j(x)$ satisfying (3.11). From this and compactness of the set D, there exist $x_1, \ldots, x_N \in D$ for some N such that

$$D \subset \bigcup_{\alpha=1}^{\alpha=N} B_{x_\alpha}(j(x_\alpha)). \tag{3.12}$$

So from (3.3),(3.11),(3.12) and Lemma 3, we have

$$\begin{aligned}|D| \leq \left|\bigcup_{\alpha=1}^{\alpha=N} B_{x_\alpha}(j(x_\alpha))\right| &\leq \sum_{k=1}^{\nu=M} |B^*_{x_{\alpha_k}}(j(x_{\alpha_k}))| \leq 2^{15}\sum_{k=1}^{\nu=M} |H_{x_{\alpha_k}}(j(x_{\alpha_k}))| \\ &\leq \frac{2^{15}}{\lambda}\sum_{k=1}^{\nu=M}\int_{H_{x_{\alpha_k}}(j(x_{\alpha_k}))} |f(y)|dy \leq \frac{2^{15}}{\lambda}\|f\|_{L^1(\mathbb{R}^2)}.\end{aligned}$$

The first inequality follows from (3.12), the second inequality from (3.9), the third from (3.3), the fourth from (3.11) and the last from (3.10). ∎

Remark: The proof of Lemma 1 is based on the increasing property of a' combined with the geometry of $H_x(j) \subset \{y : 2^j > x_1 - y_1 > 2^{j-1}\}$. If we define a parallelogram as $H'_x(j) \subset \{y : 2^j > y_1 - x_1 > 2^{j-1}\}$, we are not able to use properties such as (3.4) and (3.5), which are crucial factor for the proof of our Vitali-type covering lemma. For this reason, the case including the interval $[-r, 0]$ in (1.2) is not directly covered by Theorem 1. It will be interesting to extend our result to the maximal function defined on the full interval $[-r, r]$ in (1.2).

4. L^p estimate for $\mathfrak{M}^{glo}$

In proving the L^p boundedness of $\mathfrak{M}^{glo}$, we show that there exists a constant $C > 0$ independent of n such that for $n \leq 0$:

$$\left\| \left(\sum_{j \in \mathbb{Z}} |M_j^n f|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^2)} \leq C 2^{-|n|/8} \|f\|_{L^2(\mathbb{R}^2)} \tag{4.1}$$

$$\left\| \sup_{j \in \mathbb{Z}} |M_j^n f| \right\|_{L^p(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)} \quad \text{for } 1 < p \leq 2. \tag{4.2}$$

4.1. Proof of (4.1). Fix $\lambda \in \mathbb{R}^1$ and $n \leq 0, j \in \mathbb{Z}$. We define an operator $T_{j,n}^{\lambda}$ on $L^2(\mathbb{R}^1)$ by

$$T_{j,n}^{\lambda} f(x) = \int e^{-i\lambda a(x)(x-y)} \chi(\lambda a'(x) 2^{2j+n}) \varphi_j(x-y) f(y) dy,$$

where $x, y \in \mathbb{R}$. Then by dy_2 integration in the definition of $M_j^n f(x_1, x_2)$, we can observe that

$$M_j^n f(x_1, x_2) = [T_{j,n}^{\lambda} \widehat{f_\lambda}^2 (x_1)]^{\vee}(x_2).$$

where $^{\vee}$ is the inverse Fourier transform with respect to λ variable, and

$$\widehat{f_\lambda}^2 (x_1) = \int e^{-i\lambda x_2} f(x_1, x_2) dx_2.$$

So by applying the Plancherel theorem on the second variable we have

$$\sum_j \|M_j^n f\|_{L^2(\mathbb{R}^2)}^2 = \sum_j \int \|T_{j,n}^{\lambda} \widehat{f_\lambda}^2\|_{L^2(\mathbb{R}^1)}^2 d\lambda.$$

Thus by repetition of the Plancherel Theorem on the second variable, it suffices to show that for each fixed λ

$$\left\| \left(\sum_{j \in \mathbb{Z}} |T_{j,n}^{\lambda} f|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^1)} \leq C 2^{-|n|/8} \|f\|_{L^2(\mathbb{R}^1)}.$$

By using the Cotlar-Stein Lemma combined with the duality, we have only to check the following two estimates:

$$\|T_{j,n}^{\lambda} [T_{j,n}^{\lambda}]^*\|_{L^2(\mathbb{R}^1) \to L^2(\mathbb{R}^1)} \leq C 2^{-|n|/2}, \tag{4.3}$$

$$\|T_{j_1,n}^{\lambda} [T_{j_2,n}^{\lambda}]^*\|_{L^2(\mathbb{R}^1) \to L^2(\mathbb{R}^1)} \leq C 2^{-|j_1 - j_2|/4} \tag{4.4}$$

where $C > 0$ is independent of λ. We write

$$T^{\lambda}_{j_1,n}[T^{\lambda}_{j_2,n}]^* f(x) = \int K_{j_1,j_2}(x,z) f(z) dz,$$

where $K_{j_1,j_2}(x,z)$ is

$$\int e^{i\Psi(x,z,y,\lambda)} \chi(\lambda a'(x) 2^{2j_1+n}) \chi(\lambda a'(z) 2^{2j_2+n}) \varphi_{j_1}(x-y) \varphi_{j_2}(z-y) dy, \tag{4.5}$$

where

$$\Psi(x,z,y,\lambda) = \lambda(a(x)(x-y) - a(z)(z-y)).$$

Note that we shall use x, y and z as real numbers in the proof of (4.3) and (4.4).

Proof of (4.3). Let us now split our kernel $K_{j,j}(x,z) = K_1(x,z) + K_2(x,z)$ so that

$$K_1(x,z) = \chi_A(x,z) K_{j,j}(x,z)$$
$$K_2(x,z) = \chi_{A^c}(x,z) K_{j,j}(x,z)$$

where χ_A is a characteristic function supported on the set

$$A = \{(x,z) : |x-z| \leq 2^{-|n|/2} 2^j\}.$$

On the support of $A^c = \mathbb{R}^2 - A$, the derivative of the phase function Ψ with respect to y is bounded below such that

$$|\lambda(a(x) - a(z))| = \left|\lambda \int_z^x a'(u) du\right| \geq 2^{|n|/2} 2^{-j}. \tag{4.6}$$

Thus the integration by parts yields that

$$\int |K_2(x,z)| dx \leq C 2^{-|n|/2}, \quad \int |K_2(x,z)| dz \leq C 2^{-|n|/2}.$$

By measuring the support of A we obtain that

$$\int |K_1(x,z)| dx \leq C 2^{-|n|/2}, \quad \int |K_1(x,z)| dz \leq C 2^{-|n|/2}.$$

Hence the above four estimates prove (4.3). ∎

Proof of (4.4). Assume without loss of generality $j_1 \gg j_2$. Let us define a set B:

$$B = \{(x,z) : |a(x) - a(z)| \leq a'(x) 2^{j_1} 2^{\frac{3}{2}(j_1 - j_2)}\}.$$

Let us now split our kernel $K_{j_1,j_2}(x,z) = L_1(x,z) + L_2(x,z)$ so that

$$L_1(x,z) = \chi_B(x,z)K_{j_1,j_2}(x,z)$$
$$L_2(x,z) = \chi_{B^c}(x,z)K_{j_1,j_2}(x,z).$$

On the support of B^c, the derivative of our phase function Ψ with respect to y is

$$|\lambda(a(x) - a(z))| > |\lambda a'(x)2^{j_1}2^{\frac{3}{2}(j_1-j_2)}| \geq |2^{-j_2}\lambda a'(x)2^{2j_1}2^{\frac{j_1-j_2}{2}}|,$$

which is bigger then $2^{-j_2}2^{|n|}2^{\frac{j_1-j_2}{2}}$. So the integration by parts yields that

$$\int |L_2(x,z)|dx \leq C2^{\frac{-|j_1-j_2|}{2}},$$

$$\int |L_2(x,z)|dz \leq C2^{\frac{-|j_1-j_2|}{2}}. \tag{4.7}$$

From the fact that $|\lambda a'(x)2^{2j_1+n}| \approx |\lambda a'(z)2^{2j_2+n}| \approx 1$, we have $a'(z) \approx a'(x)2^{2(j_1-j_2)}$. Therefore, for each fixed x, we measure the support of z so that

$$|\{z : (x,z) \in B\}| \leq C\frac{a'(x)2^{j_1}2^{\frac{3}{2}(j_1-j_2)}}{a'(x)2^{2(j_1-j_2)}} = 2^{j_1}2^{-\frac{(j_1-j_2)}{2}}.$$

Using this we can estimate

$$\int |L_1(x,z)|dx \leq C,$$

$$\int |L_1(x,z)|dz \leq C2^{\frac{-|j_1-j_2|}{2}}. \tag{4.8}$$

Hence (4.4) follows from (4.7) and (4.8). ∎

4.2. Proof of (4.2). Let $I_k = \{t : 2^{k-1} < a'(t) \leq 2^k\}$ and define

$$P_k f(x_1,x_2) = \chi_{I_k}(x_1)f(x_1,x_2)$$

where χ_{I_k} is the characteristic function supported on the set I_k. Then

$$M_j^n f(x) = \sum_{k=-\infty}^{\infty} \chi_{I_{k-2j}}(x_1)P_{k-2j}M_j^n f(x) \tag{4.9}$$

where $P_{k-2j}M_j^n f(x)$ is

$$\chi_{I_{k-2j}}(x_1)\iiint \varphi_j(x_1-y_1)e^{i\lambda[x_2-y_2-a(x_1)(x_1-y_1)]}\chi(2^{2j+n}a'(x_1)\lambda)d\lambda f(y)dy.$$

On the support of the kernel,

$$2^{k-1} < a'(x_1)2^{2j} \leq 2^k \quad \text{and} \quad 2^{-n-1} < \lambda a'(x_1)2^{2j} \leq 2^{-n}.$$

So $2^{-n-k-1} < \lambda < 2^{-n-k+1}$. This implies that

$$P_{k-2j} M_j^n f(x) = P_{k-2j} M_j^n L_{n+k}^2 f(x), \tag{4.10}$$

where

$$\widehat{L_{n+k}^2 f}(\xi_1, \xi_2) = \big(\chi(2^{n+k-1}\xi_2) + \chi(2^{n+k}\xi_2) + \chi(2^{n+k+1}\xi_2)\big)\widehat{f}(\xi_1, \xi_2).$$

Thus from (4.9) and (4.10) combined with the fact $\sum_k |\chi_{I_{k-2j}}(x_1)|^2 \le 2$,

$$\sup_{j\in\mathbb{Z}} |M_j^n f(x)| \le \bigg(\sum_{k\in\mathbb{Z}} \Big|\sup_j P_{k-2j} M_j^n (L_{n+k}^2 f)(x)\Big|^2\bigg)^{1/2}. \tag{4.11}$$

By using $2^{2j} a'(x_1) \approx 2^k$ we can see that $P_{k-2j} M_j^n f(x)$ is majorized by

$$\chi_{I_{k-2j}}(x_1) \iint \varphi_j(x_1 - y_1) \frac{1}{2^{k+n}} \widehat{\psi}\Big(\frac{x_2 - y_2 - a(x_1)(x_1 - y_1)}{2^{k+n}}\Big) f(y_1, y_2) dy.$$

For each fixed k and n, we define the maximal function

$$\mathcal{N}^{n,k} f(x) = \sup_j N_j^{n,k} f(x)$$

where

$$N_j^{n,k} f(x) \quad = \quad \frac{\chi_{I_{k-2j}}(x_1)}{|R_x(j)|} \iint_{R_x(j)} |f(y_1, y_2)| dy$$

and

$$R_x(j) \quad = \quad \{y : 2^{j-1} < x_1 - y_1 \le 2^j, |x_2 - y_2 - a(x_1)(x_1 - y_1)| < 2^{k+n}\}.$$

Observe from the fact that $\widehat{\psi}$ is rapidly decreasing function,

$$\sup_k |P_{k-2j} M_j^n f(x)| \quad \le \quad C\mathcal{N}^{n,k} f(x).$$

Thus by the Littlewood-Paley inequality for the square sum of $L_{n+k}^2 f$ in (4.11), we see that it suffices to show that there exist C independent of n such that for $1 < p \le 2$:

$$\bigg\|\bigg(\sum_{k\in\mathbb{Z}} |\mathcal{N}^{n,k}(|f_k|)|^2\bigg)^{1/2}\bigg\|_{L^p(\mathbb{R}^2)} \le C \bigg\|\bigg(\sum_{k\in\mathbb{Z}} |f_k|^2\bigg)^{1/2}\bigg\|_{L^p(\mathbb{R}^2)}. \tag{4.12}$$

We shall show that for $1 < p < \infty$, there is a constant C independent of n, k such that,

$$\|\mathcal{N}^{n,k} f\|_{L^p(\mathbb{R}^2)} \le C\|f\|_{L^p(\mathbb{R}^2)}. \tag{4.13}$$

Now we show that (4.13) implies (4.12). If we have (4.13), then (4.12) follows for $p = 2$. This and the positivity of the kernel of the operator $N_j^{n,k}$ imply that

$$\|\{\mathcal{N}^{n,k} f_k\}\|_{L^2(l^\infty)} \leq C\|\{f_k\}\|_{L^2(l^\infty)}.$$

From (4.13) we have

$$\|\{\mathcal{N}^{n,k} f_k\}\|_{L^p(l^p)} \leq C\|\{f_k\}\|_{L^p(l^p)}$$

for any $p > 1$. By interpolation of above two vector valued norm space, we have (4.12) for $4/3 < p \leq 2$. Repeat this process until we obtain (4.12) for any $1 < p \leq 2$.

Now let us return to (4.13). We have from the support condition for each fixed n, k,

$$N_j^{n,k} f(x) \leq N_j^{n,k} f_{R_j}(x) \tag{4.14}$$

where $f_{R_j}(y) = \chi_{R_j}(y) f(y)$ and $R_j = \bigcup_{x_1 \in I_{2k-j}} R_x(j)$. From (4.14),

$$\|\mathcal{N}^{n,k} f\|^p_{L^p(\mathbb{R}^2)} \leq \iint \sum_{j\in\mathbb{Z}} |N_j^{n,k} f_{R_j}(x)|^p dx_1 dx_2 \leq \sum_{j\in\mathbb{Z}} \iint |f_{R_j}(x)|^p dx_1 dx_2. \tag{4.15}$$

By (4.15), we see that (4.13) is proved if we show that

$$R_{j_1} \cap R_{j_2} = \emptyset \quad \text{if} \quad |j_1 - j_2| > 10. \tag{4.16}$$

Proof of (4.16). Without loss of generality, $j_1 > j_2 + 10$. It suffices to show that for $2^{k-2j_1-1} < a'(x_1) \leq 2^{k-2j_1}$ and $2^{k-2j_2-1} < a'(z_1) \leq 2^{k-2j_2}$:

$$R_x(j_1) \cap R_z(j_2) = \emptyset.$$

Since a' is increasing, it follows that $x_1 \leq z_1$. Assume that $y \in R_x(j_1) \cap R_z(j_2)$, then

$$2^{j_1-1} < x_1 - y_1 \leq 2^{j_1} \quad \text{and} \quad 2^{j_2-1} < z_1 - y_1 \leq 2^{j_2}.$$

This means that y_1 is on the left to x_1 with distance 2^{j_1} and y_1 is on the left to z_1 with distance 2^{j_2}. This is a contradiction with the fact that $x_1 \leq z_1$ and $2^{j_1} > 2^{10} 2^{j_2}$. Hence (4.16) is proved. ∎

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